Elastic Stability of Orthotropic Conical and Cylindrical Shells Subjected to Axisymmetric Loading Conditions†

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Summary

An analysis is presented for determining the general instability load of reinforced conical and cylindrical shells. The types of loading considered in the investigation include combined axial load and rotationally symmetric pressure distributions which vary along the generator of the cone. The method of solution is based on an application of the principle of minimum potential energy of an equivalent orthotropic shell. General results are reduced to a relatively simple formula and are compared with other investigations and experimental values.

Symbols

a	= base radius of shell, in
$\bar{A}_f$	= area of circumferential frame including effective sheet, in^2
$\bar{A}_s$	= area of longitudinal stiffener including effective sheet, in^2
b	= stiffener spacing, in
d	= frame spacing, in
E_ξ, E_θ	= moduli of elasticity for orthotropic shell, psi
$G_{\xi\theta}$	= shear modulus of orthotropic shell, psi
h	= thickness of shell, in
h_f	= distributed area of frame, $\bar{A}_f/d$, in
h_s	= distributed area of stiffener, $\bar{A}_s/b$, in
$\bar{h}$	= effective shear thickness of shell wall, in
H	= $[12(2t_3 - \nu^2)/\lambda^4k^2h^2]/[8t_1t_3t_4(I_f/I)]$
$\bar{H}$	= $12(1 - \nu^2)/\lambda^4a^2h^2$
I_f	= distributed bending moment of inertia of frame, $\bar{I}_f/d$, in^3
I_s	= distributed bending moment of inertia of stiffener, $\bar{I}_s/b$, in^3
$\bar{I}_f$	= bending moment of inertia of frame including effective sheet, in^4
$\bar{I}_s$	= bending moment of inertia of stiffener including effective sheet, in^4
J	= $(J_f + J_s)/2$
J_f	= distributed torsional moment of inertia of frame, $\bar{J}_f/d$, in^3
J_s	= distributed torsional moment of inertia of stiffener, $\bar{J}_s/b$, in^3
$\bar{J}_f$	= torsional moment of inertia of frame, in^4
$\bar{J}_s$	= torsional moment of inertia of stiffener, in^4
k	= $a(1 - \beta/2)/\sin\alpha$
K	= λk
K_1	= $-[2a(1 - \beta/2)/L \sin^2\alpha] \int_0 r(\bar{N}_\xi/p) \times \cos^2\lambda\xi d\xi$
K_2	= $-[2a(1 - \beta/2)/L] \int_0^L (\bar{N}_\theta/P)(\sin^2\lambda\xi/r)d\xi$
L	= slant length of shell, in
m, n	= axial and circumferential wave numbers
$\bar{N}_\xi$, $\bar{N}_\theta$, $\bar{N}_{\xi\theta}$	= axial, circumferential, and shear stress resultants per unit length, lb/in
p	= reference pressure for a particular loading condition, psi
p_{cr}, $\bar{p}_{cr}$	= critical pressures for conical and cylindrical shells, psi
P	= $[p_{cr}K_2\, 12(1 - \nu^2)]/[2t_4\lambda^2H^{1/4}Eh^3(I_f/I)]$
$\bar{P}$	= $[\bar{p}_{cr}K_2\, 12(1 - \nu^2)]/\lambda^2\bar{H}^{1/4}Eh^3$
r	= $a[1 - \beta(\xi/L)]$
$S/2$	= $K_1/k^2K_2H^{1/4}$
$\bar{S}/2$	= $(K_1/a^2K_2 - 1)/\bar{H}^{1/4}$
u, v, w	= axial, circumferential and radial displacements of shell middle-surface
α	= base angle of shell
$\bar{\alpha}$	= pressure variation parameter defined in Ref. 9
β	= $(L/a)\cos\alpha$
γ	= ratio of axial to lateral pressure
λ	= $m\pi/L$
$\nu_{\xi\theta}$, $\nu_{\theta\xi}$	= Poisson's ratio for orthotropic shell
ξ, θ, ζ	= axial, circumferential, and radial coordinates
ϕ	= $n^2/H^{1/4}K^2\sin^2\alpha$
$\bar{\phi}$	= $[1 + (n^2/\lambda^2a^2)]/\bar{H}^{1/4}$
ϕ_0	= $\cot^2\alpha/K$

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(1) Introduction

THIS PAPER is concerned with the linear problem of buckling of reinforced cylindrical and conical shells subjected to axisymmetric loading conditions. The major purpose of the investigation is to develop criteria for the elastic stability of such shell structures and to compare the general results with preliminary experiments.

Although several analytical studies have already been performed by other investigators, the results for the most part are either limited as to the type of loading and geometry considered, or presented in a form not readily applicable for design purposes. The present paper provides for the first time a relatively simple but accurate formula for evaluating the linear problem of buckling for a broad class of shell geometries and loadings.

In the problems to be considered, the analysis is confined to the treatment of an equivalent orthotropic shell; that is, a shell whose properties differ in the axial and circumferential directions. The orthotropic treatment distributes the stiffener effects uniformly over the shell and is capable of affording a good approximation to the elastic, stiffened shell if the reinforcing is sufficiently closely spaced and if the buckled wave pattern of the deformed shell extends over a number of stiffener elements.

Admitting the possibility of elastically orthotropic behavior, an expression for the general instability load of the equivalent orthotropic shell is established by an application of energy principles and a Rayleigh-Ritz approximation. In applying this approximate method, it is necessary to determine a plausible set of displacements which accurately describe the deflected shape of the cone.

The displacements which are introduced during the course of the present investigation are similar to those used for the elastic buckling of cylindrical shells. These relationships are shown to be satisfactory for truncated cones in which the ratio of the smallest diameter of the cone to the largest diameter is greater than one-quarter.

Finally, the critical buckling pressure for a stiffened, isotropic cone is obtained by correlating the properties of the equivalent orthotropic shell to the properties of the reinforced shell. These results are compared with other investigations and preliminary experimental values for stiffened and unstiffened conical and cylindrical shells.

(2) Discussion for Conical and Cylindrical Shells

(A) *Elastic Relations of an Orthotropic Material*

A material is called "orthotropic" if the mechanical properties of the material differ in the directions of three mutually perpendicular axes. Thus, an orthotropic material has three moduli of elasticity, one for each of the principal axes, E_α, E_β, E_γ; three moduli of shearing rigidity, each associated with two of these axes, $G_{\alpha\beta}$, $G_{\beta\gamma}$, $G_{\gamma\alpha}$; and six Poisson ratios, two associated with the stress applied in the direction of each of the three axes, $\nu_{\alpha\beta}$, $\nu_{\alpha\gamma}$, $\nu_{\beta\gamma}$, $\nu_{\beta\alpha}$, $\nu_{\gamma\alpha}$, and $\nu_{\gamma\beta}$. The values of these elastic constants are not independent, but are related according to the reciprocal theorem in the following manner:

$$\left.\begin{aligned} E_\alpha \nu_{\beta\alpha} &= E_\beta \nu_{\alpha\beta} \\ E_\alpha \nu_{\gamma\alpha} &= E_\gamma \nu_{\alpha\gamma} \\ E_\beta \nu_{\gamma\beta} &= E_\gamma \nu_{\beta\gamma} \end{aligned}\right\} \qquad (1)$$

In the present investigation, we limit the discussion, however, to two principal directions since the resulting elastic properties are adequate for application to thin, reinforced shell construction. If the two principal directions (ξ, θ) are selected so as to form an orthogonal curvilinear coordinate system on the surface, the elastic constants are given by E_ξ, E_θ, $G_{\xi\theta}$, $\nu_{\xi\theta}$, $\nu_{\theta\xi}$; and Eq. (1) becomes

$$E_\xi \nu_{\theta\xi} = E_\theta \nu_{\xi\theta} \qquad (2)$$

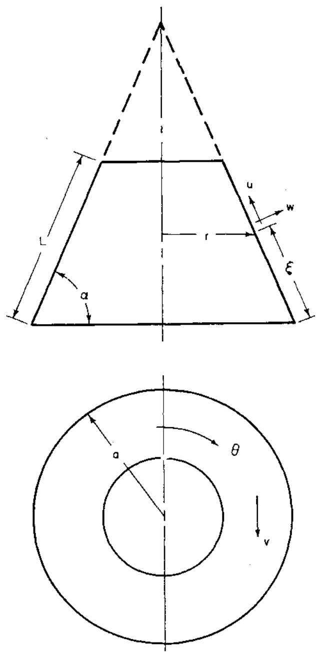

FIG. 1. Notation for a conical shell.

Hence, the elastic behavior of a thin, orthotropic shell can be adequately defined by four independent elastic properties.

(B) *Middle-Surface Strains and Displacements*

The expressions for the middle-surface strains and curvatures of a deformed conical shell are given as follows: (Fig. 1)

$$\left.\begin{aligned} \epsilon_\xi &= u_{,\xi} + 1/2\, w_{,\xi}{}^2 \\ \epsilon_\theta &= (1/r)\,(v_{,\theta} - u \cos\alpha + w \sin\alpha) + 1/2\, w_{,\theta}{}^2/r^2 \\ \epsilon_{\xi\theta} &= v_{,\xi} + u_{,\theta}/r + v\cos\alpha/r + w_{,\xi} w_{,\theta}/r \\ \chi_\xi &= -w_{,\xi\xi} \\ \chi_\theta &= -\frac{1}{r}\,(w_{,\theta\theta}/r - w_{,\xi}\cos\alpha) \\ \chi_{\xi\theta} &= -\frac{1}{r}\,(w_{,\xi\theta} + w_{,\theta}\cos\alpha/r) \end{aligned}\right\} \qquad (3)$$

The above are Donnell-type relationships for the middle-surface strains and curvatures and are identical to the expressions developed by Seide[1] with a change in the coordinate system.

The strains and curvatures are related to the force and moment resultants for a homogeneous orthotropic material by,

$$\left.\begin{aligned} N_\xi &= B_\xi(\epsilon_\xi + \nu_{\theta\xi}\epsilon_\theta) \\ N_\theta &= B_\theta(\epsilon_\theta + \nu_{\xi\theta}\epsilon_\xi) \\ N_{\xi\theta} &= N_{\theta\xi} = (G_{\xi\theta}h)\,\epsilon_{\xi\theta} = B_{\xi\theta}\epsilon_{\xi\theta} \\ M_\xi &= D_\xi(\chi_\xi + \nu_{\theta\xi}\chi_\theta) \\ M_\theta &= D_\theta(\chi_\theta + \nu_{\xi\theta}\chi_\xi) \\ M_{\xi\theta} &= -M_{\theta\xi} = -(G_{\xi\theta}h^3/6)\chi_{\xi\theta} = -D_{\xi\theta}\chi_{\xi\theta} \end{aligned}\right\} \qquad (4)$$

where, B_ξ, B_θ, D_ξ, and D_θ represent the average values of the extensional and bending rigidities in the axial and circumferential directions, respectively.

(C) Energy Relations

The potential energy for an elastic system is governed by the relation

$$V = U + \Omega \tag{5}$$

in which, U is the strain energy due to bending and stretching of the shell and Ω is the potential energy of the external force system. The shell strain energy may be expressed as the surface integral

$$U = 1/2 \int_S (N_\xi \epsilon_\xi + N_\theta \epsilon_\theta + N_{\xi\theta}\epsilon_{\xi\theta} + M_\xi \chi_\xi + M_\theta \chi_\theta - 2M_{\xi\theta}\chi_{\xi\theta})dS \tag{6}$$

In this expression, the quantities N_ξ, N_θ, $N_{\xi\theta}$ ... $M_{\xi\theta}$ are the force and moment resultants obtained by integrating the stresses σ_ξ, σ_θ, and $\sigma_{\xi\theta}$ through the shell thickness.

The potential energy of the external-force system is equal to the product of the external load and the increase in volume enclosed by the shell. Thus, in the case of hydrostatic loading (p_0), the potential energy may be expressed as

$$\Omega = p_0(V_1 - V_0) \tag{7}$$

in which, V_1, V_0 represent the volumes enclosed by the deformed and undeformed middle surface.

(D) Infinitesimal Theory of Buckling

The infinitesimal theory of buckling requires that the variation of the change in potential energy of the system with respect to the allowable displacements must be zero. Expressed mathematically

$$\delta(\Delta V) = 0 \tag{8}$$

where

$$\Delta V = \Delta U + \Delta\Omega \tag{9}$$

Assuming that the deformed shell prior to buckling is very close to its original shape, the expression, Eq. (3), for the middle-surface strains and curvatures may be considered to represent the change in strains and curvatures after buckling occurs. Hence, the displacements u, v, w, and the force and moment resultants N_ξ, N_θ ... $M_{\xi\theta}$ are now taken as the additional displacements, forces, and moments in the shell after buckling and Eqs., (6) and (7) as the changes in strain energy and potential energy of the external-force system respectively.

Since the middle-surface strains and curvatures are measured from the position of incipient buckling, the potential energy expression must be modified to include the energy stored by the shell in a compressed but unbuckled state. The term to be added to the potential energy expression is given by,

$$\Delta U_M = \int_S (\bar{N}_\xi \epsilon_\xi + \bar{N}_\theta \epsilon_\theta + \bar{N}_{\xi\theta}\epsilon_{\xi\theta} + \bar{M}_\xi \chi_\xi + \bar{M}_\theta \chi_\theta - 2\bar{M}_{\xi\theta}\chi_{\xi\theta})dS \tag{10}$$

in which, $\bar{N}_\xi$, $\bar{N}_\theta$ ... $\bar{M}_{\xi\theta}$ are the membrane stresses and moments existing in the shell prior to buckling.

The strain-energy relation (10) for the membrane stress resultants is capable of appreciable simplification if the shell is assumed under rotationally symmetric loading and initial bending stresses are neglected. Omitting the terms $\bar{N}_{\xi\theta}$, $\bar{M}_\xi$, $\bar{M}_\theta$, and $\bar{M}_{\xi\theta}$; Eq. (10) reduces to the form,

$$\Delta U_M = \int_S (\bar{N}_\xi \epsilon_\xi + \bar{N}_\theta \epsilon_\theta)dS \tag{11}$$

By substituting Eqs. (3) and (4) into Eqs. (6) and (11) and retaining all second order terms, the total expression for the changes in strain energy of the orthotropic shell becomes,

$$\begin{aligned}\Delta U_T = &\int_S \left\{\bar{N}_\xi \left(u,_\xi + \frac{1}{2} w,_\xi{}^2\right) + \bar{N}_\theta \left[\frac{1}{r}(v,_\theta - u\cos\alpha + w\sin\alpha) + \frac{1}{2} w,_\theta{}^2/r^2\right]\right\} dS + \\ &\frac{1}{2}\int_S \left\{B_\xi[u,_\xi{}^2 + 2\nu_{\theta\xi}(v,_\theta - u\cos\alpha + w\sin\alpha)(u,_\xi/r)] + B_\theta\left[\frac{1}{r}(v,_\theta - u\cos\alpha + w\sin\alpha)\right]^2 + G_{\xi\theta}h[v,_\xi + u,_\theta/r + v\cos\alpha/r]^2\right\} dS + \\ &\frac{1}{2}\int_S \left\{D_\xi[w,_{\xi\xi}{}^2 + 2\nu_{\theta\xi}(w,_{\theta\theta}/r - w,_\xi\cos\alpha)(w,_{\xi\xi}/r)] + D_\theta\left[-\frac{1}{r}(w,_{\theta\theta}/r - w,_\xi\cos\alpha)\right]^2 + G_{\xi\theta}h^3/3\left[-\frac{1}{r}(w,_{\xi\theta} + w,_\theta\cos\alpha/r)\right]^2\right\} dS\end{aligned} \tag{12}$$

TABLE 1. Evaluation of the Constants, t_i, as a Function of the Geometric Parameter β

β	t_1	t_3	t_4
0.00	0.50000	0.50000	0.50000
0.10	0.50074	0.50018	0.50027
0.20	0.50334	0.50081	0.50470
0.25	0.50554	0.50134	0.50803
0.35	0.51242	0.50300	0.51621
0.40	0.51748	0.50417	0.52532
0.50	0.53223	0.50749	0.54946
0.55	0.54273	0.50985	0.56400
0.60	0.55625	0.51276	0.58430
0.65	0.57382	0.51638	0.61134
0.70	0.59704	0.52093	0.64769
0.75	0.62853	0.52670	0.69795

The terms in the expression ($\Delta\Omega$) for the change in potential energy of the external force system are not presented herein. However, it is noted that terms in $\Delta\Omega$ which are linear in u, v, w cancel with similar terms in the strain-energy expression (12) by the principal of virtual work. Also, if the degree of approximation used in obtaining (12) is maintained, the non-linear terms in $\Delta\Omega$ are of the same order-of-magnitude as terms omitted from the strain-displacement relations and are neglected in what is to follow.

(E) Upper Bound for the Critical Buckling Pressure

An upper bound for the critical buckling pressure of the orthotropic shell will be determined by the Rayleigh-Ritz method. In applying this approximate method, it is necessary to determine a plausible set of displacements which accurately describe the deflected shape of the cone. In view of experimental observations for the unstiffened conical shell, the following set of displacements has been selected for the present investigation:

$$\begin{aligned} u &= A\cos(n\theta)\cos(m\pi\xi/L) \\ v &= B\sin(n\theta)\sin(m\pi\xi/L) \\ w &= C\cos(n\theta)\sin(m\pi\xi/L) \end{aligned} \qquad (13)$$

in which, A, B, and C are undetermined coefficients and m and n are number of lobes in the axial and circumferential directions. Substituting these equations into the simplified energy relation (12) and integrating yields a homogeneous quadratic function of A, B, and C, namely,

$$\Delta V = a_{11}A^2 + a_{12}AB + a_{13}AC + a_{22}B^2 + a_{23}BC + a_{33}C^2 \qquad (14)$$

The coefficients in this expression are defined in Appendix A.

The condition for the minimization of the potential energy of the system is obtained by setting the derivatives of Eq. (14) with respect to the unspecified coefficients equal to zero. This operation yields a set of linear homogeneous equations for the coefficients of the displacements, the determinant of which vanishes for nontrivial values of the critical load.

$$\begin{vmatrix} 2a_{11} & a_{12} & a_{13} \\ a_{12} & 2a_{22} & a_{23} \\ a_{13} & a_{23} & 2a_{33} \end{vmatrix} = 0 \qquad (15)$$

(F) Membrane State of Stress

When initial bending stresses and the influence of boundary conditions are neglected, the resultant state of stress at incipient buckling is statically determined from the resultant forces per unit length $\bar{N}_\xi$ and $\bar{N}_\theta$. These quantities are collected in terms of the non-dimensional parameters K_1, and K_2, defined by

$$K_1 = -\frac{2a(1-\beta/2)}{L\sin^2\alpha}\int_0^L r\left(\frac{\bar{N}_\xi}{p}\right)\cos^2\lambda\xi d\xi$$

$$K_2 = -\frac{2a(1-\beta/2)}{L}\int_0^L \left(\frac{\bar{N}_\theta}{p}\right)\frac{\sin^2\lambda\xi}{r}\,d\xi$$

For illustrative purposes, the resultant forces for a conical shell subjected to a vertical pressure (γp), applied uniformly over the polar cap of the small diameter of the cone, and a uniform lateral pressure (p) normal to the generator are given by

$$\left.\begin{aligned} \bar{N}_\xi &= -\frac{p}{2r\sin\alpha}\,[r^2 - a^2(1-\gamma)(1-\beta)^2] \\ \bar{N}_\theta &= -pr/\sin\alpha \end{aligned}\right\} \qquad (16)$$

Substituting Eq. (16) into the expressions for K_1, K_2 and integrating yields

$$\left.\begin{aligned} K_1 &= \\ &\frac{k^3[(1-\beta+\beta^2/3+\beta^2/2\pi^2)-(1-\gamma)(1-\beta)^2]}{2(1-\beta/2)^2} \\ K_2 &= k \end{aligned}\right\} \qquad (17)$$

where

$$\beta = (L/a)\cos\alpha, \quad k = a(1-\beta/2)/\sin\alpha$$

Other loading conditions, including rotationally symmetric pressure distributions which vary along the generator of the cone, can be treated in a similar manner.

(3) Theoretical Results for Conical and Cylindrical Shells

When the determinant, Eq. (15), is expanded, the results for the orthotropic conical shell can be approximately expressed as

$$p_{cr} = \frac{1}{k^2(\lambda^2K_1 + K_2\,n^2/\sin^2\alpha)}\,\{(D_\xi + 2t_1D_\theta\cot^2 a/K^2)K^4 + 4[t_3(\nu_{\theta\xi}D_\xi + D_{\xi\theta}) - D_\theta(t_1 - t_3)]\,K^2n^2/\sin^2\alpha + 2t_4D_\theta n^4/\sin^4\alpha + B_\theta(2t_3 - \nu_{\theta\xi}\nu_{\xi\theta})K^4/k^2\,Amn\} \qquad (18)$$

where

$$Amn = \frac{1}{k^4}\left[K^4 + \frac{B_\theta}{B_{\theta\xi}}\left(2t_3 - \nu_{\theta\xi}\nu_{\xi\theta} - 2v_{\xi\theta}\frac{B_{\theta\xi}}{B_\xi}\right)\frac{K^2n^2}{\sin^2 a} + 4t_1t_3\left(\frac{B_\theta}{B_\xi}\right)\frac{n^4}{\sin^4 a}\right]$$

In the limiting case of a cylindrical shell, the t_i's, given in Appendix A and tabulated in Table 1, are equal to one-half and Eq. (18) reduces to a form which is identical to the relationship developed in Ref. 2 for an orthotropic cylindrical shell under a uniform external pressure.

$$p_{cr} = \frac{EI_s}{k^2(\lambda^2 K_1 + K_2\, n^2/\sin^2\alpha)} \{[1 + 2t_1(I_f/I_s)(\cot^2\alpha/K^2)]K^4 + 4t_3[\nu + (1-\nu)(J/I_s) + (I_f/I_s)(1 - t_1/t_3)] \times K^2n^2/\sin^2\alpha + 2t_4(I_f/I_s)\, n^4/\sin^4\alpha + [(2t_3 - \nu^2)h_f/(1-\nu^2)I_s]K^4/k^2 A_{mn}\} \quad (19)$$

in which

$$A_{mn} = \frac{1}{k^4}\{K^4 + 2(h_f/\bar{h})[(2t_3 - \nu^2)/(1-\nu) - \nu(\bar{h}/h_s)]k^2n^2/\sin^2\alpha + 4t_1t_3(h_f/h_s)n^4/\sin^4\alpha\}$$

For the minimum value of the general instability load, it is necessary to minimize Eq. (19) with respect to the wave numbers, m and n. However, in the present investigation of reinforced shells, m is set equal to unity and n, allowed to vary continuously. This procedure assumes that the axial wave length extends over all of the stiffener elements.

In the event that the buckle pattern extends only over a certain number of the stiffening elements ($m > 1$), the minimum result can be readily obtained from the equation developed below. The procedure would consist of evaluating the critical load parameter, based on $m = 1$, for various modified lengths of the cone, and selecting that value for which p_{cr} is a minimum. Here, no inordinate difficulties are anticipated since it is reasonable to assume that the nodal points of the buckled configuration occur at the stiffening frames and a few numerical calculations for successive frame combinations should indicate the trend for the minimum result.

Also, in the dimensional range of greatest interest, defined by the moderate length cone—i.e.,

$$10^2 < \left(\frac{2t_3 - \nu^2}{8t_1t_3t_4}\right)^{1/2}(L^2/kh) < 10^4 \quad (20)$$

and for applications to frame stiffened shells, Eq. (19) for all practical purposes reduces to the form:

$$p_{cr}K_2\, 12(1-\nu^2)/E\lambda^2h^3 = \frac{1}{(K_1/k^2K_2 + n^2/K^2\sin^2\alpha)} \times \left\{2t_4\left(\frac{I_f}{I}\right)\frac{n^4}{K^4\sin^4\alpha} + \frac{12(2t_3 - \nu^2)/\lambda^4k^2h^2}{4t_1t_3n^4/K^4\sin^4\alpha}\right\} \quad K_2 \neq 0 \quad (21)$$

This expression was obtained by neglecting terms of the lowest order in $n^2/K^2\sin^2\alpha$.

Using the notation,

$$\left.\begin{aligned} P &= \frac{p_{cr}K_2\, 12(1-\nu^2)}{2t_4\lambda^2H^{1/4}Eh^3(I_f/I)} \\ \phi &= n^2/H^{1/4}K^2\sin^2\alpha \\ S/2 &= K_1/k^2K_2H^{1/4} \\ H &= \frac{12(2t_3 - \nu^2)/\lambda^4k^2h^2}{8t_1t_3t_4(I_f/I)} \end{aligned}\right\} \quad (22)$$

(A) *Conical Shells*

If the properties of the orthotropic shell are correlated to those of an equivalent reinforced shell, and use is made of the assumption that $\nu_{\xi\theta} = \nu_{\theta\xi} = \nu$; Eq. (18) becomes

Eq. (21) can be rewritten as

$$P = (\phi^2 + 1/\phi^2)/(\phi + S/2) \quad (23)$$

Then, from

$$\partial P/\partial\phi = 0 \quad (24)$$

$$\phi(\phi^4 - 3) + S(\phi^4 - 1) = 0 \quad (25)$$

The numerical values of a root of this equation can be approximated by,

$$\phi = [(3 + S)/(1 + S)]^{1/4} \quad \text{for, } \phi > 0 \quad (26)$$

The approximate solution coincides with the root of Eq. (25) in the limiting cases $S \to 0, \infty$ and is graphically compared with the real root for all values of S in Fig. 2. Note that the maximum percentage deviation is of the order of magnitude of 1 percent.

Substituting Eq. (26) into Eq. (23) the final expression for the general instability load is

$$P = \left(\frac{3+S}{1+S} + 1\right)\Big/\left(\frac{3+S}{1+S}\right)^{1/2} \times \left[\left(\frac{3+S}{1+S}\right)^{1/4} + S/2\right] \quad (27)$$

For the case of unstiffened cones, the ratio (I_f/I), collected in terms of the geometric parameter S in Eq. (27) is set equal to unity.

(B) *Cylindrical Shells*

If the cylindrical shell is reinforced by evenly-spaced circumferential frames, its stability is discussed in the same manner as that of a stiffened conical shell. For this purpose, the frames are replaced by equivalent increases in wall thickness and flexural rigidity in the circumferential direction. The resulting expression for the critical pressure parameter in this case is readily obtained from Eq. (27), with restriction (20) holding, by equating $\alpha = \pi/2$.

When the scope of the investigation is confined to unstiffened cylinders, it is possible of course to deter-

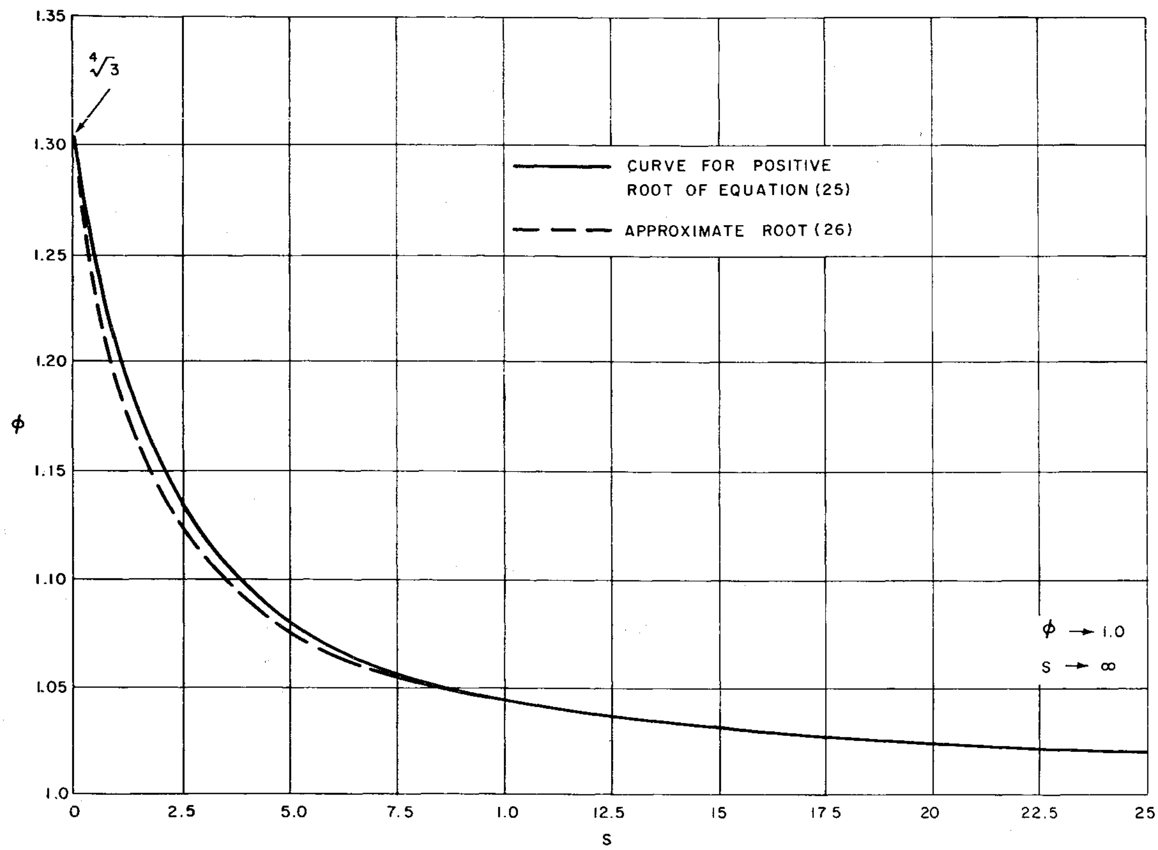

FIG. 2. Graphical comparison of the approximate root (26) for a range of the parameter $\bar{S}$.

mine the critical buckling pressure from Eq. (27) subject to condition (20). However, some interesting simple results are found for unstiffened cylinders with restriction (20) removed. In these instances, Eq. (19) is capable of further simplification and reduces to the following expression:

$$\bar{P} = [\bar{\phi}^2 + (1/\bar{\phi}^2)]/[\bar{\phi} + \bar{S}/2] \tag{28}$$

where

$$\left.\begin{aligned} \bar{P} &= \frac{\bar{p}_{cr}K_2\,12(1-\nu^2)}{\lambda^2\bar{H}^{1/4}Eh^3} \\ \bar{\phi} &= \left(1 + \frac{n^2}{\lambda^2 a^2}\right)/\bar{H}^{1/4} \\ \bar{S}/2 &= (K_1/a^2K_2 - 1)/\bar{H}^{1/4} \\ \bar{H} &= 12(1-\nu^2)/\lambda^4a^2h^2 \end{aligned}\right\} \tag{29}$$

The minimum value of the critical load can be determined by formally minimizing $\bar{p}_{cr}$ with respect to the wave numbers m and n, appearing in the above equations in terms of the parameters λ and $\bar{\phi}$. Although the procedure is straightforward, it has been previously demonstrated—i.e., Ref. 3—that the minimum value of the critical pressure for isotropic cylinders occurs on the boundary of the admissible domain formed by the constraining inequalities $m \geq 1$ and $n \geq 2$; that is, the critical pressure is a minimum either for $m = 1$ and arbitrary integer values of n, or for $n = 2$ and $m = 1, 2, 3 \ldots$.

In the case that follows, a limiting form of the solution to the above equation is evaluated for $m = 1$ and n continuously varying. The results are

$$\bar{P} = \frac{\left(\dfrac{3+\bar{S}}{1+\bar{S}} + 1\right)}{\left(\dfrac{3+\bar{S}}{1+\bar{S}}\right)^{1/2}\left[\left(\dfrac{3+\bar{S}}{1+\bar{S}}\right)^{1/4} + \bar{S}/2\right]} \quad \text{for } \bar{S} > (-1/4) \tag{30}$$

The numerical value of the critical pressure parameter for $n = 2$ and arbitrary integer values of m is not presented herein since Eqs. (3) are simplified strain-displacement relations based on the assumption of a large number of circumferential waves.

From an inspection of Eqs. (27) and (30), it is observed that the expression for the critical pressure parameter of moderately long cylinders and cones, defined by Eq. (20), is identical in form to the expression obtained for cylindrical shells with restriction (20) removed. In fact, when α is equated to $\pi/2$ in Eq. (27), the only dissimilarity in the equations appears in the parameters S and $\bar{S}$ which differ by the quantity $1/H^{1/4}$. For large values of the curvature parameter H (likewise $\bar{H}$), the expressions (27) and (30), as expected, yield similar results; however, at small values of H, suitable agreement between the equations is obtained only if the ratio of axial-to-lateral loading, defined by K_1/a^2K_2, is large when compared to unity. Thus, the predictions of Eqs. (27) and (30) are equivalent, except for small values of H and K_1/a^2K_2. In this case, the results obtained for the moderately-long shell are below those given by Eq. (30).

(4) Comparison of Present Results with Other Investigations

Eq. (27) represents an approximate but relatively accurate expression for the buckling of moderately-long cones under rotationally symmetric loading. In order to compare these results with several previous solutions, the quantities, S and I_f/I in Eq. (27), are set equal to zero and unity, respectively. These conditions reduce the application of the above equation to unstiffened conical shells subjected to a uniform lateral pressure. Also, it has been previously shown that in the dimensional ranges defined by Eq. (20), the critical stress for cylinders under a uniform lateral pressure is substantially the same as in the case of a hydrostatic pressure loading (see Batdorf[4]). A similar result is likewise expected for the moderately-long cone. Thus, from Eq. (27),

$$p_{cr}K_2\, 12(1 - \nu^2)/E\lambda^2h^3 = 1.236(L/k^{1/2}h^{1/2})[(2t_3 - \nu^2)t_4^3/t_1t_3]^{1/4} \quad (31)$$

Accordingly, the buckling pressure $\bar{p}_{cr}$, of an equivalent* cylindrical shell is given by,

$$\bar{p}_{cr}K_2\, 12(1 - \nu^2)/E\lambda^2h^3 = 1.0395(L/k^{1/2}h^{1/2})(1 - \nu^2)^{1/4} \quad (32)$$

* A cylinder which is considered as "equivalent" to a particular cone had a length equal to the slant length of the cone and a radius equal to the average radius of curvature of the cone.

and the relationship for the critical pressure ratio, $(p_{cr}/\bar{p}_{cr})$, as

$$(p_{cr}/\bar{p}_{cr})^4 = 2(t_4^3/t_1t_3)[(2t_3 - \nu^2)/(1 - \nu^2)] \quad (33)$$

This ratio is shown plotted as a function of the geometric parameter, β, in Fig. 3. The final results are compared with the more recent theoretical predictions of Seide.[5]

As shown in Fig. 3, the values of the pressure ratios given by the present theory are in close agreement with the predictions obtained by Seide for β less than three-quarters. For higher values of β, however, the present results diverge very rapidly from Seide's, giving larger buckling pressure ratios for relatively small increases in β. A possible explanation for the difference is attributed to the inadequacy of the displacements in describing the deflected shape for nearly complete cones. This inadequacy is reflected in the Rayleigh-Ritz method by a higher calculated value of total potential energy of the system and, therefore, is expected to indicate larger pressure ratios than the correct values. The contradiction for the present results occurring slightly below Seide's results for small β is attributed to the terms omitted from the investigation.

As further comparison of the present results for conical shells it is demonstrated that by equating the t_i's equal to one-half, Eq. (33) reduces to the identical relationship developed by Niordson[6] for β less than one-third. Niordson's results are somewhat remarkable in view of the many approximations used in the investigation and the simplicity and accuracy of the final presentation.

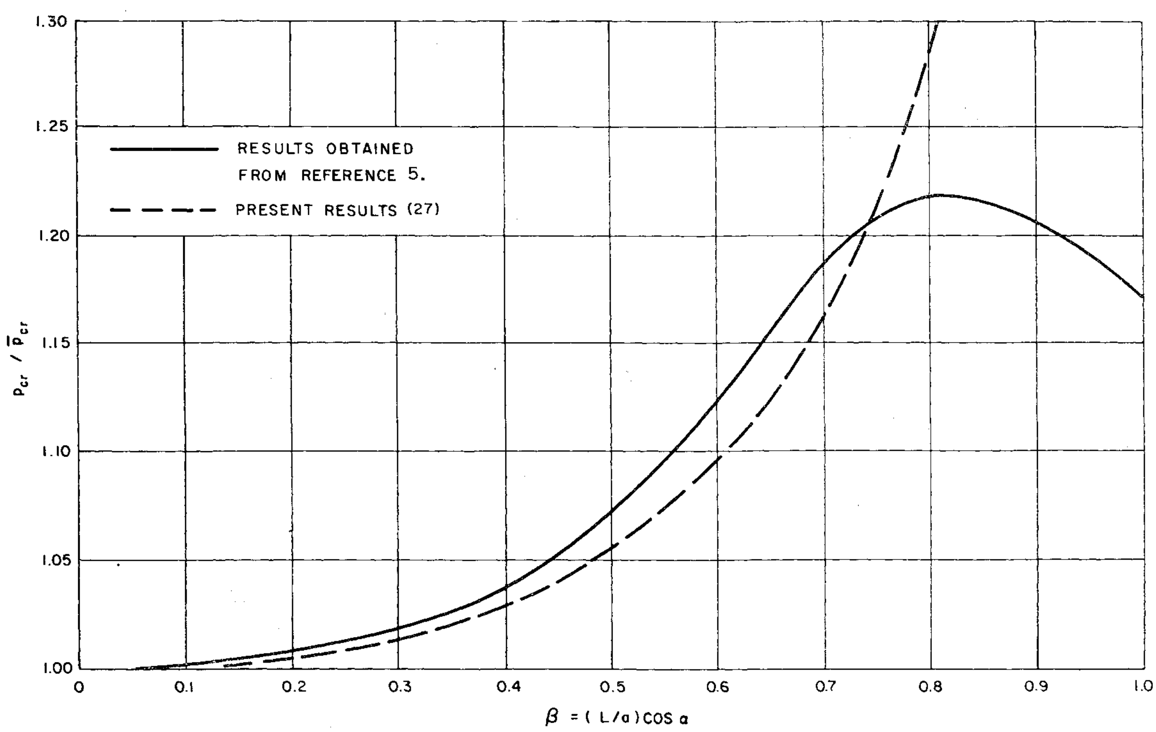

FIG. 3. Comparison of critical pressure ratio for a uniform external loading.

It is of interest to note here that the relationship (33) for the critical pressure ratio was determined on the basis of moderately-long cones and cylinders. When the ratio is calculated by means of a more nearly complete theory, Eq. (18), it is found that only a very slight discrepancy exists between the resultant values and the values obtained above. This circumstance is of primary importance since it permits rather simply the extrapolation of results for the buckling pressures of cones defined by,

$$(L^2/kh)[(2t_3 - \nu^2)/(8t_1t_3t_4)]^{1/2} < 100 \qquad (34)$$

The procedure would consist of modifying the complete results for an equivalent cylindrical shell (30) by the pressure ratio factor $(p_{cr}/\bar{p}_{cr})$ as obtained from Eq. (27).

In comparing the analytical results for cylindrical shells it is found from Eq. (27) that the numerical value for the critical load of ring-stiffened cylindrical shells under hydrostatic pressure is given by

$$\bar{p}_{cr}a\ 12(1 - \nu^2)/E\lambda^2h^3 = 1.0395[(1 - \nu^2)L^4/a^2h^2]^{1/4}(I_f/I)^{3/4} \qquad (35)$$

This result is very gratifying in that it coincides with the form which was developed by Batdorf[7] and Gerard.[8] Moreover, the effect of nonuniformity of loading on the buckling characteristics of circular cylinders can be illustrated and compared with previous analytical studies. As a typical example, consider the associated effect of a cylindrical shell subjected to a lateral pressure varying linearly in the longitudinal direction. If the remarks are confined to the moderately-long cylinder, some interesting simple results are obtained. That is the parameter $\bar{S}$ in Eq. (30) is set equal to zero; in which case, the equation becomes after an appropriate substitution for the value of K_2

$$\bar{p}_{cr}a\ 12(1 - \nu^2)/E\lambda^2h^3 = 1.0395[(1 - \nu^2)L^4/a^2h^2]^{1/4}/(1 - \bar{\alpha}/2) \qquad (36)$$

The analytical results previously developed for cylinders of moderate-length are identical with those graphically illustrated by Weingarten.[9] The comparison for the short-cylinder range is not presented herein; however, Weingarten's results can be shown to be in close agreement with Eq. (30).

(5) Comparison of Theory with Appropriate Experiment

In order to assess the applicability of the theoretical results experimental studies were conducted on a series of unstiffened cylindrical shells subjected to combined axial load and lateral pressure.* The test specimens used in the investigation were formed from extruded 6061-T6 aluminum alloy tubing. Thickness variation

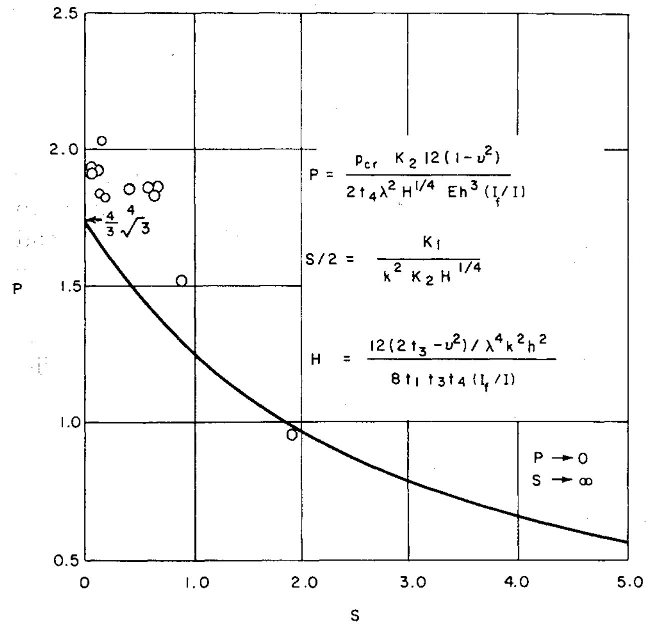

FIG. 4. Comparison of present results with preliminary experimental values.

and out-of-roundness of shell were reduced by spinning and machining the cylindrical sections. The shells were adhesively bonded and bolted to heavy steel loading flanges for testing. A description of the cylinder geometries and types of loads used in the investigation is not presented herein; but, it is mentioned that the experimental studies were conducted on moderately long shells subjected to a uniform lateral pressure loading up to ratios of axial-to-lateral pressure loadings of 30 to 1. The results are shown for comparison with the theoretical predictions in Fig. 4.

The experimental results for this series of tests are shown to be in reasonable agreement with the theoretical results for small ratios of axial-to-lateral pressure loadings, indicated in Fig. 4 by values of S approximately equal to zero. The slight discrepancy which exists is attributed to small shell irregularities and to the additional edge constraint imposed on the experimental models. The test results for slightly higher ratios of axial-to-lateral loading are also in agreement with the theoretical predictions and in the limiting case of a ratio of 30 to 1 the test result corresponds with the theoretical value.

The significant conclusion to be drawn from the present comparison is that there exists a range of shell geometries and loading conditions for which the classical predictions of buckling loads are essentially the same as those obtained from experiment. The reason for the existence of a range is attributed to the relative insensitivity of the cylinders to large-deflection effects. That is the test cylinders exhibited only a small drop in load in the post-buckling region and, as such, were relatively unaffected by initial shell imperfections.

The influence of initial imperfections on the post-buckling behavior for the cylinders tested was evident from a visual observation of the buckle pattern of the

* The experimental studies were conducted by R. Brown, R. Homewood, and J. Pierro in the Applied Mechanics Laboratory of Avco RAD.

deformed shells. When the cylinders were subjected to a uniform lateral pressure the buckle pattern consisted of full-length longitudinal buckles uniformly distributed around the shell. As the ratio of axial-to-lateral loading was increased, full-length diamond-shaped buckles occurred at the midsection of the shell. In the limiting case tested, the buckle pattern was more pronounced and consisted of relatively short diamond-shaped buckles occurring slightly below the loading flange and extending to the midheight of the cylinder.

Since very short-diamond shape buckles are characteristic of a sudden decrease of load in the post-buckling region,[10] it is apparent from the description just given that the influence of initial imperfections is becoming more predominant with corresponding increases in the ratio of axial-to-lateral pressure loading. Thus, the test data for the lateral pressure loading and to a somewhat lesser extent for the higher ratios of axial-to-lateral loading were reasonably well approximated by the classical theory.

In the case of higher axial loadings or for shell geometries which exhibit an extreme drop in load in the post-buckling region, the experimental test data is then expected to fall below the classical predictions. Consequently, the use of Fig. 4 for values of S greater than unity should not be attempted until extensive experimental studies are conducted to qualify the applicability of the classical theory for predicting the critical loads in this range.

It is briefly mentioned that the parameter S is of particular significance and provides a useful correlation parameter in the plotting of new experimental test results. This is accountable by the fact that this parameter was purposely collected in such a manner as to reflect the ratio of axial-to-lateral loading, K_1/k^2K_2, to the curvature parameter H. Thus, the implications of values of S greater than unity are either high axial loading, or a reduced axial loading but for a shorter shell geometry.

Since both of these cases characterize the area in which large-deflection effects are most pronounced, it is conjectured that an appropriate plotting of new data against this parameter may be beneficial in determining a unique cut-off point for S. In this way, it may be possible to establish an appropriate utilization range for the classical theory and, thus, provide the designer structural information required for a reliable, efficient design.

The comments, thus far, have been confined to unstiffened cylindrical shells; however, the same remarks are expected to hold true for the stiffened cylinders as well as for the unstiffened and stiffened cones. In fact in the case of a reinforced cylinder under hydrostatic pressure, the results of the present analysis—i.e., Eq. (35)—have been shown to be in agreement with the investigation by Gerard and Becker,[8] in which excellent correlation is obtained between theory and experimental data.

Moreover, it is also noted that Gerard and Becker[8] demonstrated remarkable agreement between experimental data and classical theory for stiffened cylinders under high axial loading. Although this correlation is in contrast to the agreement usually found for the isotropic cylinders, it substantiates the previous remarks concerning the existence of a unique cutoff point for the classical predictions of buckling loads. For example, if one considers the parameter S for a stiffened cylinder, it is seen to be much smaller than the corresponding S for an unstiffened cylinder of identical shape and under the same loading condition, the difference being the multiplicative factor (I/I_f). Thus, it is conjectured that while the value of S for the unstiffened cylinder may lie outside the range previously indicated for the classical linear theory, the corresponding S of the stiffened cylinder can be made to lie well within the range of validity by decreasing the factor (I/I_f)*. Experimental studies are currently being planned to confirm these remarks for the stiffened and unstiffened conical and cylindrical shells.

(6) Conclusions

This paper has demonstrated the application of the classical shell theory in determining the critical pressure of reinforced cylindrical and conical shells under combined loading. The final results have been reduced to a form which contains two nondimensional parameters; one parameter, P, essentially dependent on a critical reference pressure and the other parameter, S, determined by the combined effect of shell geometry and ratio of axial-to-lateral loading. As discussed in the test, the significance of collecting the parameters in the prescribed manner is to present a simplified approach for comparing the classical theory with experimental data for a broad range of shell geometries and loadings. In this way it is possible to develop appropriate utilization limits for the classical theory and to provide the designer structural information required for a reliable, efficient design.

In the limiting case of a uniform external pressure, the present results are shown to be in close agreement with those obtained by Seide[5] for the isotropic cone and with Bodner,[2] Batdorf,[7] and Gerard[8] for the reinforced cylinder. Also, recent experiments on unstiffiened cylinders subjected to combined loading indicate that there exists a reasonable correlation between the theoretical predictions for buckling loads and the experimental results. This correlation lends credence to the analytical results and is attributed, in part, to the extreme attention given to avoid imperfections in the fabrication and clamping of the test specimens.

It is remarked that the present analysis was restricted to the general instability characteristics of a reinforced shell; that is, a deflection pattern which extends over a number of the stiffener elements. There

* Since specific attention in this paper is given only to shells which behave orthotropically, a decrease in I/I_f corresponds to a decrease in stiffener spacing.

are, of course, a large class of important stability problems which are not characterized by this description, such as inter-bay buckling in which the facing sheet buckles within panels bounded by the stiffening system.* Although some of these problems are directly amenable to analysis by the methods of the present paper, the essential point is to inform the reader that these forms of instability are not discussed within the context of the paper and cannot be dismissed as though nonexistent.

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* A complete and comprehensive explanation of the various failure modes for reinforced shells is outlined by Becker and Gerard.[8]

Appendix A

Definition of Geometric Parameters

If a constant factor, multiplying each of the a_{ij} terms, is disregarded; the coefficients in Eq. (14) are defined as,

$$a_{11} = B_\xi K^2/2 + B_\theta t_1 K\phi_0 + B_{\xi\theta} t_1 n^2/\sin^2\alpha$$

$$a_{12} = -1/\sin\alpha[\nu_{\theta\xi}B_\xi Kn + B_\theta t_2\phi_0 n + B_{\xi\theta}n(K + \phi_0 t_2)]$$

$$a_{13} = -(\nu_{\theta\xi}B_\xi K + B_\theta t_2\phi_0)$$

$$a_{22} = B_\theta t_3\, n^2/\sin^2\alpha + B_{\xi\theta}K(K/2 + t_3\phi_C)$$

$$a_{23} = 2B_\theta t_3\, n/\sin\alpha$$

$$a_{33} = -p/2(K_1\lambda^2 + K_2\, n^2/\sin^2\alpha) + B_\theta t_3 + 1/k^2\{(D_\xi/2 + t_1 D_\theta\cot^2\alpha/K^2)K^4 + 2K^2n^2/\sin^2\alpha[t_3(\nu_{\theta\xi}D_\xi + D_{\xi\theta}) - D_\theta(t_1 - t_3)] + D_\theta t_4 n^4/\sin^4\alpha\}$$

where

$$t_1 = -\frac{[1 - (\beta/2)]}{2\beta}[\ln(1-\beta) + g_1(\beta)]$$

$$t_2 = -\frac{m\pi[1 - (\beta/2)]^2}{\beta^2}[g_2(\beta)]$$

$$t_3 = -\left[t_1 + \frac{(1-\beta/2)}{\beta}\ln(1-\beta)\right]$$

$$t_4 = -\frac{m^2\pi^2}{\beta^3}[(1-\beta/2)^3 g_1(\beta)]$$

and

$$g_1(\beta) = \cos(2m\pi/\beta)\int_{2m\pi/\beta}^{2m\pi/\beta - 2m\pi}\frac{\cos y}{y}dy + \sin(2m\pi/\beta)\int_{2m\pi/\beta}^{2m\pi/\beta - 2m\pi}\frac{\sin y}{y}dy$$

$$g_2(\beta) = \sin(2m\pi/\beta)\int_{2m\pi/\beta}^{2m\pi/\beta - 2m\pi}\frac{\cos y}{y}dy - \cos(2m\,\pi/\beta)\int_{2m\pi/\beta}^{2m\pi/\beta - 2m\pi}\frac{\sin y}{y}dy$$

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